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INERTIAL FORMS OF NAVIER–STOKES EQUATIONS ON THE SPHERE

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Abstract. The Navier–Stokes equations for incompressible flows past a two–dimensional sphere are considered in this article. The existence of an inertial form of the equations is established. Furthermore for the first time for fluid equations, we derive an upper bound on the dimension of the differential system (inertial manifold) which fully reproduces the infinite dimensional dynamics. This bound is expressed in terms of Grashof Numbers.

Introduction

Inertial manifolds are smooth finite dimensional manifolds that attract all orbits of a dissipative dynamical system at an exponential rate. By restricting the dynamical system to this manifold we obtain a finite dimensional one called the inertial form of the system. The inertial form of a system has exactly the same dynamics as the initial one, due in particular to the so–called asymptotic completeness property (cf. [CFNT], [FSTi]).

The introduction of inertial manifolds in [FST] and their name were motivated by the hope that inertial manifolds would exist for the Navier–Stokes equations (at least in dimension two) and that they would be able to describe the *inertial regimes* of turbulence. This project has been hampered by a restrictive condition for the existence of inertial manifolds which appeared in [FST] and in all subsequent works on inertial manifolds, namely the spectral gap condition. This condition is not satisfied by the Navier–Stokes equations, even in space dimension two. A breakthrough was recently made by M. Kwak

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[K1] who embedded the Navier–Stokes equations on a torus (space periodic flows) into a reaction diffusion system for which the spectral gap condition is satisfied when the ratio of the two periods is a rational number. In this case Kwak obtains, in this manner, an inertial form for the Navier–Stokes equations.

In this article we want to address the case, very important for meteorology, of the Navier–Stokes equations past the sphere. We use the same methods as in [K1] but our treatment is totally different because, in particular, we encounter here the difficulties related to the geometry. We are able as in [K1] to reduce these equations to a reaction–diffusion system for which the spectral gap condition is satisfied. Furthermore we are able in our case to derive an upper bound on the dimension of the inertial manifold in terms of the Grashof number. Although this bound is high, it is still polynomial (and not exponential) with respect to the Grashof number.

The article is organized as follows. Section 1 describes the Navier–Stokes equations on S^2. In Section 2 we introduce the embedded system and study its relation to the initial one. Absorbing sets and attractors for the embedded system are studied in Section 3. Inertial manifolds are obtained in Section 4. The dimension of the inertial manifold is estimated in Section 5. Finally we conclude in Section 6 with some complementary remarks on flows past a torus and on the barotropic equation of the atmosphere.

1. Navier-Stokes Equations on S^2

Consider the spherical coordinate system $(x^1, x^2) = (\theta, \varphi)$ on S^2, which $\theta \in (0, \pi)$ the colatitude and $\varphi \in (0, 2\pi)$ the longitude. The Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

with its inverse denoted by (g^{ij}). For geometric notations, see e.g. [A] and for, Navier-Stokes flows, Sec. 3.4, Chapt. III in [T1].

For simplicity, we use $u = u^i \frac{\partial}{\partial x^i} = (u^i)$ to denote a vector field on S^2. We can then write the 2D Navier–Stokes equations on S^2 as follows:

$$\begin{cases} u_t + \nabla_u u + \nabla p - \nu\Delta u = f, \\ \text{div } u = 0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where the covariant derivative $\nabla_u u$, the divergence div, the gradient ∇ and the Laplace–Beltrami operator Δ are defined by

$$\begin{cases} \nabla_u u = u^k u^i_{;k} \dfrac{\partial}{\partial x^i}, \\ \text{div } u = u^i_{;i}, \\ \nabla p = p_{;k} g^{ik} \dfrac{\partial}{\partial x^i}, \\ (\Delta u)^i = g^{kl} u^i_{;kl} - u^i. \end{cases} \tag{1.2}$$

As we discussed in the introduction, we consider in this article the stream function formulation of the Navier–Stokes equations. To this end, we define the curl operators for both scalar and vector functions as follows:

$$\begin{cases} \text{curl } \psi = -n \times \nabla\psi, \\ \text{curl } u = -\text{div } (n \times u), \end{cases}$$

where n is the outward normal vector, ψ is a scalar function, and u is a vector function on S^2. Then we can show that

$$\begin{cases} \nabla_u u = \nabla \left(\dfrac{u^2}{2}\right) - u \times \text{ curl } u, \\ \Delta u = \nabla(\text{div}\, u) + (u \times n)\, \text{curl } u. \end{cases} \tag{1.3}$$

Now we set

$$\begin{cases} u = \text{curl } \psi, \\ \zeta = \text{curl } u = -\Delta\psi, \end{cases} \tag{1.4}$$

where ψ is the stream function. For simplicity, let

$$A_0 = -\Delta : H_0 \to H_0, \tag{1.5}$$

be the minus Laplace–Beltrami operator on S^2 for scalar functions with inverse $A_0^{-1} : H_0 \to H_0$. The function space H_0 here and the function space V_0 used later on are defined by

$$\begin{cases} H_0 = L^2(S^2)/\mathbb{R} = \{\zeta \in L^2(S^2) | \displaystyle\int_{S^2} \zeta dS^2 = 0\}, \\ V_0 = \dot{H}^1(S^2), \end{cases} \tag{1.6}$$

where $\dot{H}^s(S^2) = H^s(S^2)/\mathbb{R}$ for any $s \in \mathbb{R}$. Then (1.4) can be written as

$$\zeta = A_0\psi,\ \psi = A_0^{-1}\zeta,\ u = \text{curl}\,(A_0^{-1}\zeta). \tag{1.7}$$

We then infer from (1.1) the following stream function formulation of the 2D Navier–Stokes equations on the sphere:

$$\begin{cases} \zeta_t + \text{div}\,[\zeta u] + \nu A_0 \zeta = F = \text{curl}\, f, \\ u = \text{curl}\,(A_0^{-1}\zeta), \\ \zeta\big|_{t=0} = \zeta_0. \end{cases} \tag{1.8}$$

Obviously, the operator A_0 is an unbounded, self–adjoint linear operator with compact inverse, and with domain $D(A_0) = \dot{H}^2(S^2)$. Moreover, using the spectrum of A_0, we can define its power $A_0^s, s \in \mathbb{R}$ and

$$A_0^s : \dot{H}^{s+2} \to \dot{H}^s,\ \forall s \in \mathbb{R}$$

is an isomorphism.

2. The Embedded System

2.1. The Equations for $\nabla\zeta$ and ζu.

Let $\zeta_0 \in D(A_0)$ and $\zeta(t)$ be a solution of (1.5). We define an injection map $J : V_0 \to H$ by

$$J(\zeta) = (\zeta, \tilde{v}, \tilde{w}) \tag{2.1}$$

where

$$\begin{cases} \tilde{v} = \nabla\zeta, \\ \tilde{w} = \zeta \operatorname{curl}(A_0^{-1}\zeta) = \zeta u. \end{cases}$$

The Hilbert space H used here and the Hilbert space V which will be used later, are defined by (cf. Remark 2.1 below)

$$\begin{cases} H = H_0 \times H_1 \times H_2, \\ H_1 = \nabla H^1(S^2), H_2 = L^2(TS^2), \\ V = V_0 \times V_1 \times V_2, \\ V_1 = \nabla H^2(S^2), V_2 = H^1(TS^2). \end{cases} \tag{2.2}$$

According to Hodge decomposition theorem (cf. [A]), we have

$$C^\infty(TS^2) = \{\nabla\phi \mid \phi \in C^\infty(S^2)\} \oplus \{\operatorname{curl}\psi \mid \psi \in C^\infty(S^2)\},$$

which implies that

$$H^s(TS^2) = \nabla H^{s+1}(S^2) \oplus \operatorname{curl} H^{s+1}(S^2), \ \forall s \geq 0. \tag{2.3}$$

Therefore, H_1 and V_1 in (2.2) are well–defined.

Moreover, for simplicity, we need to define two linear unbounded operators A_i on H_i $(i = 1, 2)$ as follows:

$$\begin{cases} A_1 = -\Delta : H_1 \to H_1, \\ A_2 = -\Delta : H_2 \to H_2, \end{cases}$$

the operator A_1 being the restriction of $-\Delta$ in $H_1 = \nabla H^1(S^2)$ with domain $D(A_1) = \nabla H^3(S^2)$, and the operator A_2 being the minus Laplacian $-\Delta$ in $H_2 = L^2(TS^2)$ with domain $D(A_2) = H^2(TS^2)$.

Then we have

Lemma 2.1. *The functions $J(\zeta)$ defined by (2.1) satisfies the following system of equations*

$$\begin{cases} \zeta_t + \operatorname{div}\tilde{w} + \nu A_0\zeta = F, \\ \tilde{v}_t + \nabla(\operatorname{div}\tilde{w}) + \nu A_1\tilde{v} = \nabla F, \\ \tilde{w}_t + \nu A_2\tilde{w} = Fu + \zeta f - 2\nu Tr[\tilde{v} \otimes \nabla u] - (\tilde{v}\cdot u)u - \zeta \operatorname{curl}[A_0^{-1}(\tilde{v}\cdot u)], \end{cases} \tag{2.4}$$

where $u = \mathrm{curl}\,(A_0^{-1}\zeta)$.

Proof. Obviously, $(2.4)_1$ is the same as (1.8). Applying the gradient operator to $(2.4)_1$, we obtain easily $(2.4)_2$.

By definition, and using (1.8) and $u_t = \mathrm{curl}\,(A_0^{-1}\zeta_t)$, we find

$$\begin{aligned}\tilde{w}_t &= \zeta_t u + \zeta u_t \\ &= \{F - \mathrm{div}\,[\zeta u] - \nu A_0\zeta\}u + \zeta\,\mathrm{curl}\,\{A_0^{-1}[F - \mathrm{div}\,[\zeta u] - \nu A_0\zeta]\}.\end{aligned}$$

We then infer from the identity

$$\Delta(av) = (\Delta a)v + a\Delta v + 2Tr((\nabla a)\otimes\nabla v),$$

that

$$\Delta\tilde{w} = (\Delta\zeta)u + \zeta\Delta u + 2Tr((\Delta\zeta)\otimes\nabla u).$$

Therefore,

$$\begin{aligned}&\tilde{w}_t - \nu\Delta\tilde{w} \\ &\quad = Fu + \zeta f - u\,\mathrm{div}\,(\zeta u) - \zeta\ \mathrm{curl}\,[A_0^{-1}(\mathrm{div}\,(\zeta u))] - 2\nu Tr[(\nabla\zeta)\otimes(\nabla u)],\end{aligned}$$

which implies $(2.4)_3$, using $\mathrm{div}\,(\zeta u) = \zeta\,\mathrm{div}\,u + (\nabla\zeta)\cdot u = \tilde{v}\cdot u$. The proof is complete. □

2.2. The Embeded System.

Equations (2.4) by themselves do not seem to be dissipative. Therefore in order to capture the dynamics of the NSE (1.1) or (1.5), we shall modify (2.3) in such a way that the resulting system satisfies the following properties:

(a) The dynamics on the section $J(V_0)$ is unchanged,

(b) The modified system is dissipative, and

(c) All the solutions of the modified system will converge to the section $J(V_0)$ as $t\to\infty$.

More previsely, we propose the following modified system of equations:

$$(2.5)\qquad \begin{cases}\zeta_t + \nu A_0\zeta + \mathrm{div}\ w = F,\\ v_t + \nu A_1 v + \nabla(\mathrm{div}\ w) = \nabla F,\\ w_t + \nu A_2 w - Fu - \zeta f + (v\cdot u)u + \zeta\ \mathrm{curl}\ [A_0^{-1}(v\cdot u)]\\ \qquad + 2\nu Tr[v\otimes\nabla u] + \underline{k\nu[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}](w-\tilde{w})} = 0,\end{cases}$$

where $\tilde{w} = \zeta\ \mathrm{curl}\ (A_0^{-1}\zeta)$. The system (2.5) is obtained from (2.4) by adding the underlined term; the positive constant k will be specified later on. Generally speaking, $\tilde{v}\neq v,\ \tilde{w}\neq w(\tilde{v} = \nabla\zeta, \tilde{w}$ as before).

For simplicity, we write $Z = (\zeta, v, w)$. Then we have the following theorems about the existence and properties of solutions of the embedded system (2.5).

Theorem 2.1. *We assume that F is given in H_0, and $Z_0 = (\zeta, v_0, w_0)$ is given in $V_0 \times H_1 \times H_2$. Then there exists a unique solution $Z = Z(t)$ of* (2.5) *with initial value $Z(0) = Z_0$, such that*

$$Z \in L^2(0,T; D(A_0) \times V_1 \times V_2) \cap C([0,T]; V_0 \times H_1 \times H_2), \forall\, T > 0. \tag{2.6}$$

Moreover, if $f \in V_0$ and $Z_0 \in D(A_0) \times V_1 \times V_2$, then

$$\begin{cases} Z \in L^2(0,T; D(A_0^{3/2}) \times D(A_1) \times D(A_2)) \cap C([0,T]; D(A_0) \times V_1 \times V_2), \\ Z_t \in L^2(0,T; V_1 \times H_1 \times H_2). \end{cases} \tag{2.7}$$

Theorem 2.2. *Let $\zeta(t)$ be a solution of* (1.8) *with initial value $\zeta(0) = \zeta_0 \in D(A_0)$, and let $f \in V_0$. Then $Z(t) = J(\zeta(t))$ is a solution of* (2.5) *with initial value $Z(0) = J(z_0)$ for all $t > 0$. Conversely, if $Z(t) = (\zeta(t), v(t), w(t))$ is a solution of* (2.5) *with $Z(0) = J(\zeta_0) \in D(A_0) \times V_1 \times V_2$, then $\zeta(t)$ is a solution of* (1.8) *for $t \geq 0$ and $Z(t) = J(\zeta(t))$. In particular, $J(D(A_0) \times V_1 \times V_2)$ is a positively invariant set for* (2.5).

Theorem 2.3. *For any $Z_0 = (\zeta_0, v_0, w_0) \in V_0 \times H_1 \times H_2$,*

$$|v(t) - \tilde{v}(t)|^2 + \nu^{-2}|w(t) - \tilde{w}(t)|^2 \leq e^{-\nu t}(|v_0 - \tilde{v}_0|^2 + \nu^{-2}|w_0 - \tilde{w}_0|^2), \ \forall\, t > 0. \tag{2.8}$$

Consequently,

$$\lim_{t \to \infty} |Z(t) - J(\zeta(t))| = 0.$$

The proof of Theorem 2.2 is easy. We only have to prove Theorems 2.1 and 2.3.

Proof of Theorem 2.1. First of all, we solve the equations $(2.5)_1$ and (2.9)-(2.10) below for the unknown functions $(\zeta, v - \tilde{v}, w - \tilde{w})$, by Galerkin method. Then we obtain the solution $Z = (\zeta, v, w)$ of (2.5). Since the Galerkin method is a standard procedure, here we only present some formal *a priori* estimates, which will also be used later for the existence of attractors of the embedded system (2.5).

Step 1. Equations for $\zeta, v - \tilde{v}$ and $w - \tilde{w}$. We infer from $(2.5)_1$ that

$$\tilde{v}_t + \nu A_1 \tilde{v} + \nabla(\text{div } w) = \nabla F,$$

which implies that

$$(v - \tilde{v})_t + \nu A_1(v - \tilde{v}) = 0. \tag{2.9}$$

We also have, as in the proof of Lemma 2.1 and since div $\tilde{w} = \tilde{v} \cdot u$,

$$\tilde{w}_t + \nu A_2 \tilde{w} - Fu - \zeta f - 2\nu Tr[\tilde{v} \otimes \nabla u] - (\text{div } w)u - \zeta \text{ curl } [A_0^{-1}(\text{div } w)] = 0.$$

Therefore

$$\begin{aligned}(w-\tilde{w})_t &+ \nu A_2(w-\tilde{w}) \\ &+ (v\cdot u - \text{div } w)u + \zeta \text{ curl } [A_0^{-1}(v\cdot u - \text{div } w)] \\ &+ 2\nu Tr[(v-\tilde{v})\otimes \nabla u] + k\nu[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}](w-\tilde{w}) = 0. \end{aligned} \tag{2.10}$$

Step 2. Energy estimates I. We infer from $(2.5)_1$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\zeta|^2 + \nu|A_0^{1/2}\zeta|^2 &= (F,\zeta) - (\text{div } w, \zeta) \\ &\le |F|\cdot|\zeta| + |(\text{div } (w-\tilde{w}),\zeta)| + |(\text{div } \tilde{w},\zeta)| \\ &\le |F|\cdot|\zeta| + |w-\tilde{w}|\cdot|A_0^{1/2}\zeta| \\ &\le \frac{c}{\nu}(|F|^2 + |w-\tilde{w}|^2) + \frac{\nu}{2}|A_0^{1/2}\zeta|^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}|\zeta|^2 + \nu|A_0^{1/2}\zeta|^2 \le \frac{c}{\nu}(|F|^2 + |w-\tilde{w}|^2). \tag{2.11}$$

By (2.9), we have

$$\frac{d}{dt}|v-\tilde{v}|^2 + 2\nu|A_1^{1/2}(v-\tilde{v})|^2 = 0. \tag{2.12}$$

Moreover, we deduce from (2.10) that

$$\begin{aligned}\frac{\alpha}{2}\frac{d}{dt}&|w-\tilde{w}|^2 + \nu\alpha|A_2^{1/2}(w-\tilde{w})|^2 + k\nu\alpha[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}]\cdot|w-\tilde{w}|^2 \\ &= -\alpha((v\cdot u - \text{div } w)u + \zeta \text{ curl } [A_0^{1/2}(v\cdot u - \text{div } w)] \\ &\quad + 2\nu Tr[(v-\tilde{v})\otimes\nabla u], w-\tilde{w}),\end{aligned} \tag{2.13}$$

where we introduce for convenience a constant $\alpha = \nu^{-2} > 0$.

The 1^{st} term in the right hand-side of (2.13) is equal to

$$\begin{aligned}-\alpha([(v&-\tilde{v})\cdot u + \text{div } (\tilde{w}-w)]u, w-\tilde{w}) \\ &\le c\alpha\|u\|^2_{L^8}\cdot|w-\tilde{w}|\cdot|A_1^{1/2}(v-\tilde{v})| + c\alpha\|u\|_{L^\infty}\cdot|w-\tilde{w}|\cdot|A_2^{1/2}(w-\tilde{w})| \\ &\le (\text{since } \|\zeta\|_{H^s}\sim\|\Delta\psi\|_{H^s}\sim\|\psi\|_{H^{s+2}}\sim\|\nabla\psi\|_{H^{s+1}}) \\ &\le \frac{c}{\nu}(\alpha^2|\zeta|^4 + \alpha\|\zeta\|^2_{H^{1/2}})\cdot|w-\tilde{w}|^2 + \frac{\nu}{4}|A_1^{1/2}(v-\tilde{v})|^2 + \frac{\nu\alpha}{4}|A_2^{1/2}(w-\tilde{w})|^2.\end{aligned}$$

The 2^{nd} term in the right hand side of (2.13) is equal to

$$-\alpha(\zeta\,\mathrm{curl}\,\{A_0^{-1}[(v-\tilde{v})\cdot u+\mathrm{div}\,(\tilde{w}-w)]\}, w-\tilde{w})$$
$$\leq c\alpha[\|\zeta\|^2_{H^{1/2}}|A_1^{1/2}(v-\tilde{v})|+\|\zeta\|_{H^{1/2}}|A_2^{1/2}(w-\tilde{w})|]\cdot|w-\tilde{w}|.$$

The 3^{rd} term in the right hand side of (2.13) is majorized by

$$c\nu\alpha|w-\tilde{w}|\cdot\|\zeta\|_{H^{1/2}}|A_1^{1/2}(v-\tilde{v})|.$$

Therefore

$$\begin{aligned}\alpha\frac{d}{dt}|w-\tilde{w}|^2+\nu\alpha|A_2^{1/2}(w-\tilde{w})|^2+2k\nu^{-1}[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}]\cdot|w-\tilde{w}|^2\\ \leq c\nu^{-1}(1+\nu^{-4}\|\zeta\|^4_{H^{1/2}})\cdot|w-\tilde{w}|^2+\nu|A_1^{1/2}(v-\tilde{v})|^2.\end{aligned}\tag{2.14}$$

Then, with the constant $\alpha=\nu^{-2}$ still at our disposal, we have

$$\begin{aligned}&\frac{d}{dt}[|\zeta|^2+|v-\tilde{v}|^2+\alpha|w-\tilde{w}|^2]+\nu[|A_0^{1/2}\zeta|^2+|A_1^{1/2}(v-\tilde{v})|^2+\alpha|A_2^{1/2}(w-\tilde{w})|^2]\\ &\qquad+2k\nu^{-1}[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}]\cdot|w-\tilde{w}|^2\\ &\quad\leq c\nu^{-1}(1+\nu^{-1}\|\zeta\|^4_{H^{1/2}})\cdot|w-\tilde{w}|^2+\frac{c}{\nu}|F|^2.\end{aligned}$$

Hence for k (absolute constant) large enough, we obtain

$$\begin{aligned}&\frac{d}{dt}[|\zeta|^2+|v-\tilde{v}|^2+\alpha|w-\tilde{w}|^2]+\nu[|A_0^{1/2}\zeta|^2+|A_1^{1/2}(v-\tilde{v})|^2\\ &\qquad+\alpha|A_2^{1/2}(w-\tilde{w})|^2]+k\nu^{-1}[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}]\cdot|w-\tilde{w}|^2\\ &\quad\leq\frac{c}{\nu}|F|^2.\end{aligned}\tag{2.15}$$

Furthermore we consider the inner product between $(2.5)_1$ and $A_0\zeta$, and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_0^{1/2}\zeta|^2+\nu|A_0\zeta|^2&=(F,A_0\zeta)-(\mathrm{div}\,w,A_0\zeta)\\ &\leq|F|\cdot|A_0\zeta|+|(\mathrm{div}\,(w-\tilde{w}),A_0\zeta)|+|(\mathrm{div}\,\tilde{w},A_0\zeta)|\\ &\leq\frac{c}{\nu}|F|^2+\frac{c_1}{2\nu}|A_2^{1/2}(w-\tilde{w})|^2+\frac{\nu}{4}|A_0\zeta|^2+|\mathrm{div}\,\tilde{w}|\cdot|A_0\zeta|.\end{aligned}$$

Since

$$|\mathrm{div}\,\tilde{w}|\cdot|A_0\zeta|\leq\frac{c}{\nu^3}|\zeta|^4\cdot|A_0^{1/2}\zeta|^2+\frac{\nu}{4}|A_0\zeta|^2,$$

we obtain

$$\frac{d}{dt}|A_0^{1/2}\zeta|^2 + \nu|A_0\zeta|^2 \le \frac{c}{\nu}|F|^2 + \frac{c_1}{\nu}|A_2^{1/2}(w-\tilde{w})|^2 + \frac{c}{\nu^3}|\zeta|^4 \cdot |A_0^{1/2}\zeta|^2. \tag{2.16}$$

Finally the combination of (2.12), (2.14) and (2.16) shows that

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2c_1}|A_0^{1/2}\zeta|^2 + |v-\tilde{v}|^2 + \alpha|w-\tilde{w}|^2\right] \\
&\quad + \nu\left[\frac{1}{2c_1}|A_0\zeta|^2 + |A_1^{1/2}(v-\tilde{v})|^2 + \frac{\alpha}{2}|A_2^{1/2}(w-\tilde{w})|^2\right] \\
&\qquad + 2k\nu^{-1}[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}] \cdot |w-\tilde{w}|^2 \\
&\quad \le c\nu^{-1}(1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}) \cdot |w-\tilde{w}|^2 + \frac{c}{\nu^3}|\zeta|^4 \cdot |A_0^{1/2}\zeta|^2 + \frac{c}{\nu}|F|^2,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2c_1}|A_0^{1/2}\zeta|^2 + |v-\tilde{v}|^2 + \alpha|w-\tilde{w}|^2\right] \\
&\quad + \nu\left[\frac{1}{2c_1}|A_0\zeta|^2 + |A_1^{1/2}(v-\tilde{v})|^2 + \frac{\alpha}{2}|A_2^{1/2}(w-\tilde{w})|^2\right] \\
&\quad + k\nu^{-1}[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}] \cdot |w-\tilde{w}|^2 \\
&\quad \le \frac{c}{\nu^3}|\zeta|^4 \cdot |A_0^{1/2}\zeta|^2 + \frac{c}{\nu}|F|^2.
\end{aligned} \tag{2.17}$$

By Gronwall's inequality, we obtain from (2.15) and (2.17) that

$$(\zeta, v-\tilde{v}, w-\tilde{w}) \in L^2(0,T; D(A_0)\times V_1 \times V_2) \cap L^\infty(0,T; V_0 \times H_1 \times H_2),\ \forall\, T>0. \tag{2.18}$$

Then from equation $(2.5)_1$ and (2.9)–(2.10), we can obtain *a priori* estimates for

$$(\zeta, v-\tilde{v}, w-\tilde{w})_t \in L^2(0,T; H_0 \times V_1' \times V_2'),\ \forall\, T>0.$$

Hence

$$(\zeta, v-\tilde{v}, w-\tilde{w}) \in C([0,T]; V_0 \times H_1 \times H_2),\ \forall\, T>0. \tag{2.19}$$

On the other hand, by definition, we can prove that

$$\begin{cases}
|\tilde{v}| = |\nabla\zeta| \le |A_0^{1/2}\zeta|, \\
|A_1^{1/2}\tilde{v}| \le |A_0\zeta|, \\
|\tilde{w}| = |\zeta \text{ curl } (A_0^{-1}\zeta)| \le |A_0^{1/2}\zeta|^2, \\
|A_2^{1/2}\tilde{w}| \le |A_0\zeta| \cdot |A_0^{1/2}\zeta|.
\end{cases} \tag{2.20}$$

Then the combination of (2.18)–(2.20) proves the existence of a solution $Z = (\eta, v, w)$ of (2.5) satisfying (2.6).

Step 3. Energy estimates II. From $(2.5)_1$, we also have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A_0\zeta|^2 + \nu|A_0^{3/2}\zeta|^2 &= (F, A_0^2\zeta) - (\text{div } w, A_0^2\zeta)\\ &\le |A_0^{1/2}F|\cdot|A_0^{3/2}\zeta| + |(\text{div } (w-\tilde{w}), A_0^2\zeta)| + |(\text{div } \tilde{w}, A_0^2\zeta)|\\ &\le \{|A_0^{1/2}F| + |\nabla(\text{div } (w-\tilde{w}))| + |\nabla(\nabla\zeta\cdot \text{curl } (A_0^{-1}\zeta))|\}\cdot|A_0^{3/2}\zeta|\\ &\le \frac{c}{\nu}|A_0^{1/2}F|^2 + \frac{c_2}{2\nu}|A_2(w-\tilde{w})|^2 + \frac{c}{\nu}\|\zeta\|_{H^{1/2}}^2\cdot|A_0\zeta|^2 + \frac{\nu}{2}|A_0^{3/2}\zeta|^2,\end{aligned}$$

i.e.

$$\frac{d}{dt}|A_0\zeta|^2 + \nu|A_0^{3/2}\zeta|^2 \le \frac{c}{\nu}|A_0^{1/2}F|^2 + \frac{c_2}{\nu}|A_2(w-\tilde{w})|^2 + \frac{c}{\nu}\|\zeta\|_{H^{1/2}}^2\cdot|A_0\zeta|^2. \tag{2.21}$$

By (2.9), we have

$$\frac{d}{dt}|A_1^{1/2}(v-\tilde{v})|^2 + 2\nu|A_1(v-\tilde{v})|^2 = 0. \tag{2.22}$$

Moreover from (2.10) we obtain that

(2.23)
$$\begin{aligned}&\frac{\alpha}{2}\frac{d}{dt}|A_2^{1/2}(w-\tilde{w})|^2 + \nu\alpha|A_2(w-\tilde{w})|^2 + k\nu^{-1}[1+\nu^{-4}\|\zeta\|_{H^{1/2}}^4]\cdot|A_2^{1/2}(w-\tilde{w})|^2\\ &\quad= -\alpha((v\cdot u - \text{div } w)u + \zeta \text{ curl } [A_0^{-1}(v\cdot u - \text{div } w)]\\ &\qquad + 2\nu Tr[(v-\tilde{v})\otimes\nabla u], A_2(w-\tilde{w})).\end{aligned}$$

the 1^{st} term in the right hand side of (2.23) is equal to

$$\begin{aligned}&-\alpha([(v-\tilde{v})\cdot u + \text{div } (\tilde{w}-w)]u, A_2(w-\tilde{w}))\\ &\le c\alpha\{|\zeta|^2\cdot|A_1^{1/2}(v-\tilde{v})| + \|\zeta\|_{H^{1/2}}\cdot|A_2^{1/2}(w-\tilde{w})|\}\cdot|A_2(w-\tilde{w})|.\end{aligned}$$

The 2^{nd} term in the right hand side of (2.23) is equal to

$$\begin{aligned}&-\alpha(\zeta \text{ curl } \{A_0^{-1}[(v-\tilde{v})\cdot u + \text{div } (\tilde{w}-w)]\}, A_2(w-\tilde{w}))\\ &\le c\alpha[\|\zeta\|_{H^{1/2}}|\zeta|\cdot|A_1^{1/2}(v-\tilde{v})| + \|\zeta\|_{H^{1/2}}|w-\tilde{w}|]\cdot|A_2(w-\tilde{w})|.\end{aligned}$$

The 3^{rd} term in the right hand side of (2.23) is majorized by

$$c\nu^{-1}|A_2(w-\tilde{w})|\cdot\|\zeta\|_{H^{1/2}}|A_1^{1/2}(v-\tilde{v})|.$$

Therefore

(2.24)
$$\alpha\frac{d}{dt}|A_2^{1/2}(w-\tilde{w})|^2+\nu\alpha|A_2(w-\tilde{w})|^2+k\nu^{-1}[1+\nu^{-4}\|\zeta\|_{H^{1/2}}^4]\cdot|A_2^{1/2}(w-\tilde{w})|^2$$
$$\leq\frac{c\alpha}{\nu}(|\zeta|^4+\|\zeta\|_{H^{1/2}}^2\cdot|\zeta|^2+\nu^2\|\zeta\|_{H^{1/2}}^2)\cdot|A_1^{1/2}(v-\tilde{v})|^2$$
$$+\frac{c\alpha}{\nu}\|\zeta\|_{H^{1/2}}^2|A_2^{1/2}(w-\tilde{w})|^2.$$

Finally (2.21)–(2.22) and (2.24) imply that

$$\begin{aligned}\frac{d}{dt}&\left[\frac{1}{2c_2}|A_0\zeta|^2+|A_1^{1/2}(v-\tilde{v})|^2+\alpha|A_2^{1/2}(w-\tilde{w})|^2\right]\\&+\nu\left[\frac{1}{2c_2}|A_0^{3/2}\zeta|^2+2|A_1(v-\tilde{v})|^2+\frac{\alpha}{2}|A_2(w-\tilde{w})|^2\right]\\&+2k\nu^{-1}[1+\nu^{-4}\|\zeta\|_{H^{1/2}}^4]\cdot|A_2^{1/2}(w-\tilde{w})|^2\\&\leq\frac{c}{\nu}|A_0^{1/2}F|^2+\frac{c}{\nu}[\alpha|\zeta|^4+\alpha\|\zeta\|_{H^{1/2}}^2\cdot|\zeta|^2+\|\zeta\|_{H^{1/2}}^2]\\&\quad\cdot\left[\frac{1}{2c_2}|A_0\zeta|^2+|A_1^{1/2}(v-\tilde{v})|^2+\alpha|A_2^{1/2}(w-\tilde{w})|^2\right].\end{aligned}\tag{2.25}$$

Using Gronwall's inequality as in Step 2, we can prove that if $Z_0\in D(A_0)\times V_1\times V_2$, then Z satisfies (2.7).

The proof of the uniqueness of solution is easy, and is omitted.

Theorem 2.1 is proved. □

Proof of Theorem 2.3. We infer from (2.12) and (2.14) that

$$\begin{aligned}\frac{d}{dt}[|v-\tilde{v}|^2+\alpha|w-\tilde{w}|^2]&\leq-\nu[|A_1^{1/2}(v-\tilde{v})^2+\alpha|A_2^{1/2}(w-\tilde{w})|^2]\\&\leq-\nu[|v-\tilde{v}|^2+\alpha|w-\tilde{w}|^2],\end{aligned}$$

which proves (2.8). □

3. Absorbing Sets and Attractors

First of all, Theorem 2.1 indicates that we can define the operators

$$\Sigma(t):Z_0\longrightarrow Z(t),\quad\forall\ t\geq0.\tag{3.1}$$

These operators enjoy the standard semigroup properties, and they are continuous operators in $V_0 \times H_1 \times H_2$ and in $D(A_0) \times V_1 \times V_2$.

We now prove the existence of an absorbing set of the semigroup $\{\Sigma(t)\}_{t\geq 0}$ in $V_0 \times H_1 \times H_2$. T othis end, by Gronwall's inequality, we infer from (2.15) that

$$|\zeta(t)|^2 + |(v-\tilde{v})(t)|^2 + \alpha|(w-\tilde{w})(t)|^2 \leq c|F|^2 + e^{-\nu t}(|\zeta_0|^2 + |v_0 - \tilde{v}_0|^2 + \alpha|w_0 - \tilde{w}_0|^2),$$

for any $t \geq 0$. Then for any $R > 0$, if $Z_0 = (\zeta_0, v_0, w_0) \in B_{V_0\times H_1\times H_2}(0, R)$, the ball in $V_0 \times H_1 \times H_2$ centered at 0 with radius R, there exists $t_0 = t_0(R) > 0$ such that for any $t \geq t_0$

$$\begin{cases} |\zeta(t)|^2 + |(v-\tilde{v})(t)|^2 + \alpha|(w-\tilde{w})(t)|^2 \leq c|F|^2, \\ \displaystyle\int_t^{t+1} [|A_0^{1/2}\zeta(t)|^2 + |A_1^{1/2}(v-\tilde{v})(t)|^2 + \alpha|A_2^{1/2}(w-\tilde{w})(t)|^2] dt \leq c|F|^2. \end{cases} \tag{3.2}$$

Now we return to (2.17). By the uniform Gronwall lemma (cf. Lemma 1.1 in [T1]), we obtain that

$$|A_0^{1/2}\zeta(t)|^2 + |(v-\tilde{v})(t)|^2 + |(w-\tilde{w})(t)|^2 \leq c|F|^2 \exp(c|F|^4), \ \forall\, t \geq t_0 + 1. \tag{3.3}$$

By (2.20), (3.3) shows that

$$|A_0^{1/2}\zeta(t)|^2 + |v(t)|^2 + |w(t)|^2 \leq c|f|^2 \exp(c|F|^4) + c|F|^4 \exp(c|F|^4), \ \forall\, t \geq t_0 + 1. \tag{3.4}$$

Now we define the right hand side of (3.4) as ρ_0^2, then we conclude that

$$\Sigma(t) B_{V_0\times H_1\times H_2}(0, R) \subseteq B_{V_0\times H_1\times H_2}(0, \rho_0), \ \forall\, t \geq t_0 + 1. \tag{3.5}$$

In other words, $B_{V_0\times H_1\times H_2}(0, \rho_0)$ is an absorbing set in $V_0 \times H_1 \times H_2$ for the semigroup $\Sigma(t)$.

Moreover, from (2.17), we also have

$$\begin{aligned} \int_t^{t+1} [|A_0\zeta(t)|^2 + |A_1^{1/2}(v-\tilde{v})(t)|^2 + \alpha|A_2^{1/2}(w-\tilde{w})(t)|^2] dt \\ \leq c|F|^2 + c\rho_0^2(1 + |F|^4), \ \forall\, t \geq t_0 + 1. \end{aligned} \tag{3.6}$$

Now we want to establish the existence of an absorbing set in $D(A_0) \times V_1 \times V_2$. We apply the uniform Gronwall lemma to (2.25). Then we obtain

$$|A_0\zeta(t)|^2 + |A_1^{1/2}(v-\tilde{v})(t)|^2 + \alpha|A_2^{1/2}(w-\tilde{w})(t)|^2 \leq c_1(|A_0^{1/2}F|), \ \forall\, t \geq t_0 + 2, \tag{3.7}$$

where $c_1(|A_0^{1/2}F|)$ is a function of $|A_0^{1/2}F|$. On the other hand, it is obvious that

$$\begin{cases} |A_1^{1/2}\tilde{v}| \leq c|A_0\zeta|, \\ |A_2^{1/2}\tilde{w}| \leq c|A_0\zeta|^2. \end{cases} \tag{3.8}$$

Therefore

$$|A_0\zeta(t)|^2 + |A_1^{1/2}v(t)|^2 + \alpha|A_2^{1/2}w(t)|^2 \leq c_2(|A_0^{1/2}F|), \ \forall\ t \geq t_0 + 2. \tag{3.9}$$

where $c_2(|A_0^{1/2}F|)$ is a function of $|A_0^{1/2}F|$. This shows that the ball $B_{D(A_0)\times V_1\times V_2}(0, \rho_1)$ in $D(A_0)\times V_1\times V_2$ is an absorbing set in $D(A_0)\times V_1\times V_2$, and that the operators $\Sigma(t)$ are uniformly compact for t large. Here $\rho_1^2 = c_2(|A_0^{1/2}F|)$. Hence by Theorem 1.1, Chapter I of [T1] we promptly obtain that

Theorem 3.1. *There exists a unique global attractor $\mathcal{A} \subseteq D(A_0) \times V_1 \times V_2$ of the embedded system* (2.5)*, such that $\mathcal{A}$ is compact, connected and maximal in $V_0 \times H_1 \times H_2$ and $D(A_0) \times V_1 \times V_2$.*

The following theorem shows that $\mathcal{A}$ is exactly the lifting of the attractor $\mathcal{A}_0$ of the original Navier–Stokes equations (1.1) or (1.5), via the injection map $J : V_0 \to V_0 \times H_1 \times H_2$ defined by (2.2).

Theorem 3.2. *We have*

$$\mathcal{A} = J(\mathcal{A}_0). \tag{3.10}$$

4. Inertial Manifolds

4.1. Preliminaries.

The embedded system can be rewritten in the following concise form

$$\frac{dZ}{dt} + AZ + R(Z) = 0, \tag{4.1}$$

where the linear and nonlinear operators A, R are defined by

$$AZ = \begin{pmatrix} \nu A_0\zeta + \text{div } w \\ \nu A_1 v + \nabla(\text{div } w) \\ \nu A_2 w \end{pmatrix}, \quad RZ = \begin{pmatrix} R_0(Z) \\ R_1(Z) \\ R_2(Z) \end{pmatrix}, \tag{4.2}$$

where $R_0(Z) = -F, R_1(Z) = -\nabla F$ and

$$R_2(Z) = -Fu - \zeta f + (v\cdot u)u + \zeta \text{ curl } [A_0^{-1}(v\cdot u)] + 2\nu Tr[v\otimes\nabla u] + k\nu[1+\nu^{-4}\|\zeta\|^4_{H^{1/2}}](w-\tilde{w}).$$

Then we can obtain easily that $D(A) = D(A_0)\times D(A_1)\times D(A_2)$, and $D(A)$ is dense in H.

For technical reasons, we need to use the following equivalent norm $|\cdot|_H$ in H:

$$|Z|_H \doteq (|\zeta|^2 + |v|^2 + \alpha|w|^2)^{1/2},$$

the corresponding inner product being denoted by $(\cdot,\cdot)_H$; here $\alpha = 1/\nu^2$. Then we have the following Garding inequality

$$(AZ, Z)_H \geq \frac{\nu}{6}\{|A_0^{1/2}\zeta|^2 + |A_1^{1/2}v|^2 + \alpha|A_2^{1/2}w|^2\}, \tag{4.3}$$

which can be proved easily by taking the following inequality into consideration:

$$(\nabla(\text{div } w), v) = (w, \nabla(\text{div } v)) = (w, A_1, v) \leq \frac{1}{2}|A_1^{1/2}v|^2 + \frac{1}{2}|A_2^{1/2}w|^2.$$

Moreover, as in [K1], we can prove that A is a sectorial operator, and $A^{-1}: H\to H$ is a compact linear operator. Therefore the spectrum $\sigma(A)$ consists of a countable number of eigenvalues with no finite accumulation points and each eigenvalue has finite multiplicity. Indeed we have

Lemma 4.1 (1). *The operator A has only the eigenvalues $\nu n(n+1)(n=1,2,\dots)$.*

(2). *The eigenvalue $\nu n(n+1)$ has at most multiplicity $8(2n+1)$.*

Proof. (1). We can easily prove that if λ is an eigenvalue of A, then λ has to be an eigenvalue of either νA_0 or νA_1, or νA_2. Then it suffices to prove that all the eigenvalues of A_0, A_1 and A_2 are the numbers $\{n(n+1)|n=1,2,\dots\}$ and count multiplicities. According to [CH], the eigenvalues of A_0 are the numbers $\{n(n+1)|n=1,2,\dots\}$, and the eigenfunctions corresponding to $n(n+1)$ are the $2n+1$ spherical harmonics, denoted $Y_{n,h}(h=0,\pm1,\dots,\pm n)$. Thanks to the Hodge decomposition (2.4), we can easily prove that $\nabla Y_{n,h}(h=0,\pm1,\dots,\pm n)$ are eigenvectors of A_1 corresponding to the eigenvalue $n(n+1)$, and that $\{\nabla Y_{n,h}|h=0,\pm1,\dots,\pm n, n=1,2,\dots\}$ is a complete basis of H_1. Therefore, the operator A_1 has only eigenvalues $\{n(n+1)|n=1,2,\dots\}$.

Similarly, we can also prove that the operator A_2 has only eigenvalues $\{n(n+1)|n=1,2,\dots\}$ with corresponding eigenvectors $\{\nabla Y_{n,h} + \text{curl } Y_{n,h'}|h,h'=0,\pm1,\dots,\pm n\}$.

(2). Set $\lambda = \nu n(n+1)$, and

$$A_\lambda = A - \lambda I = \begin{pmatrix} A_{0\lambda} & & B_0 \\ & A_{1\lambda} & B_1 \\ & & A_{2\lambda} \end{pmatrix},$$

where $A_{i\lambda} = \nu A_i - \lambda I (i = 0, 1, 2), B_0 = \text{div}$ and $B_1 = \nabla(\text{div})$ and all unwritten elements of the matrix vanish. Then by direct computation, we can obtain that

$$A_\lambda^k = \begin{pmatrix} A_{0\lambda}^k & & B_0^{(k)} \\ & A_{1\lambda}^k & B_1^{(k)} \\ & & A_{2\lambda}^k \end{pmatrix},$$

where $B_i^{(k)} = A_{i\lambda} B_i^{(k-1)} + B_i A_{2\lambda}^{k-1}, i = 0, 1$.

Now we suppose that $A_\lambda^k Z = 0$ for $Z = (\zeta, v, w)$. It follows immediately that $A_{2\lambda}^k w = 0$ in which case $w = 0$ or λ is an eigenvalue of A_2 and w an associated eigenvector. In the first case we infer that $A_{0\lambda}^k \zeta = 0, A_{1\lambda}^k v = 0$; hence λ is one of the numbers $\nu n(n+1)$ and ζ and/or v are associated eigenvectors of A_0, A_1. In the second case λ is again one of the numbers $\nu n(n+1)$ and w is an associated eigenvector of A_2; $w = \nabla Y_1 + \text{curl } Y_2, Y_1$ and Y_2 being linear combinations of the $Y_{n,h}(h = 0, \pm 1, \dots, \pm n)$. We infer that

$$\begin{aligned} B_0^{(2)} w &= A_{0\lambda} B_0 w + B_0 A_{2\lambda} w \\ &= A_{0\lambda} \text{div}[\nabla Y_1 + \text{curl } Y_2] \\ &= A_{0\lambda} \text{div } \nabla Y_1 \\ &= A_{0\lambda} (\triangle Y_1) \\ &= -n(n+1) A_{0\lambda} Y_1 \\ &= 0. \end{aligned}$$

Similarly, $B_1^{(2)} w = 0$ and by iteration $B_i^{(k)} w = 0$, for $i = 0, 1$ and all $k \geq 2$. Hence we obtain that

$$\ker A_\lambda^2 = \{(\zeta, v, w) \mid A_{0\lambda} \zeta = 0, A_{1\lambda} v = 0, A_{2\lambda} w = 0\},$$

and this is the generalized eigenspace of A with respect to λ. In summary, λ has at most the multiplicity $2(2n + 1 + 2n + 1 + 2(2n + 1)) = 8(2n + 1)$.

The proof is complete. □

To prove the existence of inertial manifold for the embedded system, it is common to truncate the nonlinear terms $R(Z)$. To this end, we need

Lemma 4.2. *If $F \in D(A_0)$, then $\mathcal{A}$ is bounded in $D(A)$.*

Proof. It is easy to see that if $F \in D(A_0)$, then $\mathcal{A}_0$ is bounded in $D(A_0^{3/2})$. Then Theorem 3.2 implies that $\mathcal{A}$ is bounded in $D(A)$. □

Now let ρ be a positive number such that

$$|AZ|_H \leq \frac{\rho}{2}, \ \forall \ Z \in \mathcal{A}.$$

We consider the following prepared equations of the embedded system:

$$\frac{dZ}{dt} + AZ + R_\eta(Z) = 0. \tag{4.4}$$

Here

$$R_\eta(Z) = \eta_\rho(|AZ|_H)R(Z),\ \forall\ Z \in D(A), \tag{4.5}$$

where $\eta_\rho(s) = \eta(s/\rho)$ and η is a C^∞ function from $\mathbb{R}_+$ into $[0,1]$ such that

$$\eta(s) = \begin{cases} 1 & \text{for } 0 \le s \le 1, \\ 0 & \text{for } s \ge 2, \end{cases}$$

and $\sup_{s\ge 0} |\eta'(s)| \le 2$.

Lemma 4.3. *R_η is a globally bounded operator from $D(A)$ into itself:*

$$\sup_{Z\in D(A)} |AR_\eta(Z)|_H \le \sup_{|AZ|_H \le 2\rho} |AR(Z)|_H = M_1. \tag{4.6}$$

Lemma 4.4. *R_η is a globally Lipschitz operator from $D(A)$ ito itself:*

$$|A(R_\eta(Z_1) - R_\eta(Z_2))|_H \le M_2|A(Z_1 - Z_2)|_H. \tag{4.7}$$

These lemmas are easy. Explicit values of M_1 and M_2 are needed and given below.

4.2. Inertial Manifolds.

Now we let

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N \le \lambda_{N+1} \le \ldots$$

denote the eigenvalues of A repeated according to their multipliticity. Let $P = P_N$ be the projection associated with the eigenvalues $\{\lambda_n | n \le N\}$, and $Q = Q_N = I - P$. Then PH is the direct sum of the eigenspaces corresponding to the eigenvalues $\lambda_n \le \lambda_N$, and QH is the direct sum of the eigenspaces corresponding to the eigenvalues $\lambda_n \ge \lambda_{N+1}$. It is easy to see that

$$PH \subset D(A), \quad H = PH \oplus QH, \tag{4.8}$$

and PH, QH are invariant under A.

The inertial manifold for equation (4.4) is obtained in the form $\mathcal{M} = \text{Graph}\Phi$, i.e. as the graph of a Lipschitz mapping

$$\Phi : PH \to QH \cap D(A), \tag{4.9}$$

such that

$$
\begin{cases}
\text{supp}\Phi \subset \{y \in PH, |Ay|_H \leq 2\rho\}, \\
|A\Phi(y)|_H \leq \nu^2 b, \ \forall y \in PH, \\
|A\Phi(y_1) - A\Phi(y_2)|_H \leq l|A(y_1 - y_2)|_H, \ \forall y_1, y_2 \in PH.
\end{cases}
\tag{4.10}
$$

Theorem 4.1. *We can choose N such that there exists an inertial manifold for the prepared equation (*4.4*) of the embedded system* (4.1)*, which is the graph of a Lipschitz mapping* (4.9) *satisfying* (4.10)*.*

Proof. By Lemma 4.1, it is easy to see that there is an integer $N \geq 1$ such that

$$
\begin{cases}
\lambda_{N+1} - \lambda_N = 2\nu(n+1) \geq 2M_2 \dfrac{1+l}{l}, \\
\lambda_{N+1} = \nu(n+1)(n+2) \geq \dfrac{M_1}{\nu^2 b}.
\end{cases}
\tag{4.11}
$$

Then the result can be obtained by applying the theory of inertial manifolds for nonself–adjoint operators to (4.4) (cf. [DT] and [SY]). The estimates (conditions) in (4.11) are those of [DT]; ℓ, b are arbitrarily chosen, e.g. $\ell = b = 1/2$. □

Finally we have

Theorem 4.2 *For $f \in D(A_0)$, the essential long–time dynamics of the $2D$ Navier–Stokes equations on the sphere S^2 is completely described by the system of ordinary differential equations*

$$
\frac{dy}{dt} + Ay = PF(y + \Phi(y)), \quad in \ \ PH,
\tag{4.12}
$$

where Φ is the Lipschitz map (4.9) *obtained in Theorem 4.1 satisfying* (4.10)*.*

5. Dimension of the Inertial Manifolds

In order to estimate the dimension of the inertial manifold for the embedded system given by Theorem 4.1, we need to estimate the constants M_1 and M_2 in (4.11) and the relationship between N and n in (4.11). To this end, we have

Lemma 5.1. *Suppose that N is the largest number such that $\lambda_N = \nu n(n+1)$, then*

$$
N \leq 8n(n+2).
\tag{5.1}
$$

Proof. By counting the multiplicities, we obtain that

$$
N \leq 8[(2 \cdot 1 + 1) + \cdots + (2 \cdot n + 1)] = 8n(n+2).
$$

□

Lemma 5.2. *The dimension N of the inertial manifold satisfies*

$$N \le 1 + \max\left\{ \frac{8M_1}{\nu^3 b}, \frac{8M_2^2(1+l)^2}{\nu^2 l^2} \right\}. \tag{5.2}$$

Proof. By Lemma 5.1, we obtain that

$$N \le 8(n+1)^2,$$

which implies that

$$n \ge \sqrt{\frac{N}{8}} - 1.$$

Therefore

$$\lambda_{N+1} - \lambda_N = 2\nu(n+1) \ge \nu\sqrt{\frac{N}{2}}.$$

Hence if we choose N such that

$$N \ge \frac{8M_2^2(1+l)^2}{\nu^2 l^2},$$

then $(4.11)_1$ is satisfied. Moreover, it is easy to see that if we choose N such that

$$N \ge \frac{8M_1}{\nu^3 b},$$

then $(4.11)_2$ is satisfied.

The proof is complete. □

To estimate M_1 and M_2 we need to estimate ρ as follows.

Lemma 5.3.

$$|AZ| \le \rho = c\nu^2[G_0 G_4 + G_0^{12}], \ \forall\ Z \in \mathcal{A}, \tag{5.3}$$

where G_0 and G_4 are generalized Grashof numbers[1] *defined by*

$$G_0 = \frac{|f|}{\nu^2}, \quad G_4 = \frac{\|f\|_{H^4}}{\nu^2}. \tag{5.4}$$

[1]No length appears in the definition (5.4) of G_0 and G_4 because we assumed that the radius a of the sphere is equal to one. For $a \ne 1$, we would define G_0 and G_4 by

$$G_0 = \frac{|f|a^2}{\nu^2}, \quad G_4 = \frac{\|f\|_{H^4}a^6}{\nu^2}.$$

Proof. *Step 1.* We estimate the bound for $\zeta \in \mathcal{A}_0$ in $D(A_0^{3/4})$. To this end, we have from (1.8) that

$$\frac{1}{2}\frac{d}{dt}|\zeta|^2 + \nu|A_0^{1/2}\zeta|^2 = (F,\zeta) \le \frac{c|f|^2}{\nu} + \frac{\nu}{2}|A_0^{1/2}\zeta|^2,$$

which implies that

$$\frac{d}{dt}|\zeta|^2 + \nu|A_0^{1/2}\zeta|^2 \le \frac{c|f|^2}{\nu}. \tag{5.5}$$

By Gronwall's inequality, we obtain that

$$|\zeta(t)|^2 \le \frac{c|f|^2}{\nu^2}, \tag{5.6}$$

for t large enough. Since the attractor $\mathcal{A}_0$ is time invariant, by shifting the time t, (5.6) holds true for any $t \ge 0$.

Now we derive as in [T3] (see Sec. 4) the higher order norm estimates:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|A_0^{3/2}\zeta|^2 + \nu|A_0^2\zeta|^2\\
&\quad = (F, A_0^3\zeta) - (\nabla\zeta\cdot u, A_0^3\zeta)\\
&\quad \le |A_0F|\cdot|A_0^2\zeta| + |(D^2(\nabla\zeta)\cdot u + D(\nabla\zeta)\cdot Du + D(\nabla\zeta)\cdot D^2u, A_0^2\zeta)|\\
&\quad \le |A_0^2\zeta|\cdot\{\|f\|_{H^3} + \|\zeta\|_{H^{7/2}}\cdot|\zeta| + \|\zeta\|_{H^{5/2}}\cdot\|\zeta\|_{H^{1/2}} + \|\zeta\|^2_{H^{3/2}}\}\\
&\quad \le |A_0^2\zeta|\cdot\left\{\|f\|_{H^3} + |A_0^2\zeta|^{7/8}\cdot|\zeta|^{9/8} + |A_0^2\zeta|^{6/8}\cdot|\zeta|^{10/8}\right\}\\
&\quad \le \frac{\nu}{2}|A_0^2\zeta|^2 + \frac{c\|f\|^2_{H^3}}{\nu} + \frac{c|\zeta|^{18}}{\nu^{15}},
\end{aligned}$$

i.e.

$$\frac{d}{dt}|A_0^{3/2}\zeta|^2 + \nu|A_0^2\zeta|^2 \le \frac{c\|f\|^2_{H^3}}{\nu} + \frac{c|\zeta|^{18}}{\nu^{15}} \le c\nu^3[G_3^2 + G_0^{18}], \tag{5.7}$$

where G_3 is the generalized Grashof number defined by

$$G_3 = \frac{\|f\|_{H^3}}{\nu^3} \;(\le CG_0^{1/4}G_4^{3/4} \quad \text{by interpolation}). \tag{5.8}$$

Using Gronwall's inequality, we can show that

$$|A_0^{3/2}\zeta(t)|^2 \le c\nu^2[G_3^2 + G_0^{18}], \quad \text{for } \; t \text{ large enough.}$$

Therefore

$$|A_0^{3/2}\zeta|^2 \le c\nu^2[G_3^2 + G_0^{18}],\ \forall\ \zeta \in \mathcal{A}_0. \tag{5.9}$$

Step 2. We estimate the bounds of $\mathcal{A}_0$ in $D(A_0^{1/2})$ and $D(A_0)$.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|A_0^{1/2}\zeta|^2 + \nu|A_0\zeta|^2 \\
&\quad = (F, A_0\zeta) - (\nabla\zeta\cdot u, A_0\zeta) \\
&\quad \le c|A_0\zeta|\cdot\{\|f\|_{H^1} + \|\zeta\|_{H^{3/2}}\cdot|\zeta|\} \\
&\quad \le c|A_0\zeta|\cdot\|f\|_{H^1} + c|A_0\zeta|^{7/4}\cdot|\zeta|^{5/4} \\
&\quad \le \frac{\nu}{2}|A_0\zeta|^2 + \frac{c\|f\|_{H^1}^2}{\nu} + \frac{c|\zeta|^{10}}{\nu^7},
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{d}{dt}|A_0^{1/2}\zeta|^2 + \nu|A_0\zeta|^2 &\le c\nu^3\left[\frac{\|f\|_{H^1}^2}{\nu^4} + G_0^{10}\right] \\
&\le c\nu^3\left[G_0^{4/3}G_3^{2/3} + G_0^{10}\right].
\end{aligned}$$

It follows that

$$|A_0^{1/2}\zeta(t)|^2 \le c\nu^2\left[G_0^{4/3}G_3^{2/3} + G_0^{10}\right],\quad \text{for } t \text{ large}.$$

Therefore

$$|A_0^{1/2}\zeta|^2 \le c\nu^2\left[G_0^{4/3}G_3^{2/3} + G_0^{10}\right],\ \forall\ \zeta \in \mathcal{A}_0. \tag{5.10}$$

Moreover, we also have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|A_0\zeta|^2 + \nu|A_0^{3/2}\zeta|^2 \\
&\quad \le |A_0^{1/2}F|\cdot|A_0^{3/2}\zeta| + |(D^2\zeta\cdot u + \nabla\zeta\cdot Du, A_0^{3/2}\zeta)| \\
&\quad \le |A_0^{3/2}\zeta|\cdot\{\|f\|_{H^2} + \|\zeta\|_{H^{5/2}}\cdot|\zeta| + \|\zeta\|_{H^{3/2}}\cdot\|\zeta\|_{H^{1/2}}\} \\
&\quad \le |A_0^{3/2}\zeta|\cdot\left\{\|f\|_{H^2} + |A_0^2\zeta|^{6/5}\cdot|\zeta|^{7/6}\right\} \\
&\quad \le \frac{\nu}{2}|A_0^{3/2}\zeta|^2 + \frac{c\|f\|_{H^2}^2}{\nu} + \frac{c|\zeta|^{14}}{\nu^{11}}.
\end{aligned}$$

Therefore, we obtain that

$$\frac{d}{dt}|A_0\zeta|^2 + \nu|A_0^{3/2}\zeta|^2 \le c\nu^3\left[G_0^{2/3}G_3^{4/3} + G_0^{14}\right].$$

Hence by Gronwall's inequality, we obtain as before that

$$|A_0\zeta|^2 \le c\nu^2 \left[G_0^{2/3}G_3^{4/3} + G_0^{14}\right],\ \forall\ \zeta \in \mathcal{A}_0. \tag{5.11}$$

Step 3. Now we estimate the bound for $\mathcal{A} = J(\mathcal{A}_0)$. To this end, for any $Z = (\zeta, v, w) \in \mathcal{A}$, we have $v = \nabla\zeta$ and $w = \zeta$ curl $(A_0^{-1}\zeta)$. It follows that

$$|A_1 v|^2 \le c\nu^2[G_3^2 + G_0^{18}], \tag{5.12}$$

$$\begin{aligned} |A_2 w|^2 &\le c\|\zeta\|_{H^2}^2 \cdot \|\zeta\|_{H^1}^2 \\ &\le (\text{by } (5.10) - (5.11)) \\ &\le c\nu^4[G_0^{2/3}G_3^{4/3} + G_0^{14}] \cdot [G_0^{4/3}G_3^{2/3} + G_0^{10}]. \end{aligned} \tag{5.13}$$

Then we can easily obtain using (5.8) that

$$\nu^2\{|A_0\zeta|^2 + |A_1\nu|^2 + \alpha|A_2 w|^2\} \le c\nu^4[G_0G_4 + G_0^{12}]^2,$$

which shows that

$$|AZ|_H \le c\nu^2[G_0G_4 + G_0^{12}],\ \forall\ Z \in \mathcal{A}. \tag{5.14}$$

The proof is complete. □

Now we need to estimate M_1 and M_2.
Lemma 5.4.

$$M_1 \le c\nu^3[G_0^6G_4^6 + G_0^{72}]. \tag{5.15}$$

Proof. For any $Z = (\zeta, v, w) \in D(A)$ such that $|AZ|_H \le 2\rho$, we have

$$\begin{aligned} |AR(Z)|_H^2 \le\ & 2\nu^2\{|A_0R_0(Z)|^2 + |A_1R_1(Z)|^2 + \alpha|A_2R_2(Z)|^2\} \\ & + 2|\text{div } R_2(Z)|^2 + 2|\nabla \text{ div } R_2(Z)|^2 \\ \le\ & 2\nu^2\{\|f\|_{H^3}^2 + \|f\|_{H^4}^2\} + c|A_2R_2(Z)|^2 \\ \le\ & 2\nu^2\{\|f\|_{H^3}^2 + \|f\|_{H^4}^2\} + c\{\|f\|_{H^3}^2\|\zeta\|_{H^1}^2 + \|f\|_{H^2}^2\|\zeta\|_{H^2}^2 + \|v\|_{H^2}^2\|\zeta\|_{H^2}^4 \\ & + \nu^2\|v\|_{H^2}^2\|\zeta\|_{H^2}^2 + \nu^2(1 + \nu^{-4}\|\zeta\|_{H^{1/2}}^4)^2\|w - \tilde{w}\|_{H^2}^2\} \\ \le\ & 2\nu^2\{\|f\|_{H^3}^2 + \|f\|_{H^4}^2\} + c\Big\{\|f\|_{H^3}^2\frac{\rho^2}{\nu^2} + \|f\|_{H^2}^2\frac{\rho^2}{\nu^2} + \frac{\rho^6}{\nu^6} \\ & + \nu^2(1 + v^{-8}\rho^4)^2(\rho^2 + \frac{\rho^4}{\nu^4})\Big\} \\ \le\ & (\text{with } (5.3)) \\ \le\ & c\nu^6[G_0^6G_4^6 + G_0^{72}]^2. \end{aligned}$$

Therefore, by (4.6) we obtain (5.15). □

Lemma 5.5.

$$M_2 \le c\nu[G_0^5 G_0^5 + G_0^{60}]. \tag{5.16}$$

Proof. As shown in [T1] (see (2.19) in Chapter VIII), M_2 is given by

$$M_2 = \frac{2M_1}{\rho} + c_{2\rho}, \tag{5.17}$$

where $c_{2\rho}$ is the local Lipschitz constant of R defined by

$$|A(R(Z_1) - R(Z_2))|_H \le c_{2\rho}|A(Z_1 - Z_2)|_H,\ \forall\ Z_i \in D(A), |AZ_i|_H \le 2\rho, i = 1, 2. \tag{5.18}$$

Obviously,

$$\frac{2M_1}{\rho} \le c\nu[G_0^5 G_4^5 + G_0^{60}]. \tag{5.19}$$

Now for any $Z_i \in D(A),\ |AZ_i|_H \le 2\rho, i = 1, 2$, we have

$$\begin{aligned}
|A(R(Z_1) - R(Z_2))|_H &\le |\text{div } (R_2(Z_1) - R_2(Z_2))| \\
&\qquad + |\nabla \text{ div } (R_2(Z_1) - R_2(Z_2))| + |A_2(R_2(Z_1) - R_2(Z_2))| \\
&\le (\text{since } A_2 : D(A_2) \to H_2 \text{ is an isomorphism }) \\
&\le c|A_2(R_2(Z_1) - R_2(Z_2))| \\
&\le |A(Z_1 - Z_2)|_H \cdot \left[\frac{\|f\|_{H^3}}{\nu} + \frac{\rho^3}{\nu^3} + \frac{\rho}{\nu}\right] \\
&\qquad + c\nu(1 + \nu^{-8}\rho^4)|A_2(w_1 - \tilde{w}_1 - (w_2 - \tilde{w}_2))| \\
&\qquad + c\nu^{-3}(|A_2 w_1| + |A_2\tilde{w}_1|)(\|\zeta_1\|_{H^{1/2}}^4 - \|\zeta_2\|_{H^{1/2}}^4) \\
&\le |A(Z_1 - Z_2)|_H \cdot \Big[\nu G_3 + \frac{\rho^2}{\nu^3} + c\nu(1 + \nu^{-8}\rho^4) \\
&\qquad\quad + \nu^{-1}(1 + \nu^{-8}\rho^4)\rho + \frac{\rho^4}{\nu^7} + \frac{\rho^5}{\nu^9}\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
c_{2\rho} &\le \left[\nu G_3 + \frac{\rho^2}{\nu^3} + c\nu(1 + \nu^{-8}\rho^4) + \nu^{-1}(1 - \nu^{-8}\rho^4)\rho + \frac{\rho^4}{\nu^7} + \frac{\rho^5}{\nu^9}\right] \\
&\le c\nu[G_0^5 G_0^5 + G_0^{60}].
\end{aligned} \tag{5.20}$$

Then the combination of (5.17) and (5.19)–(5.20) proves (5.16). □

We are in position to state the main theorem in this section, which is now a direct consequence of Lemmas 5.2 and 5.4 – 5.5.

Theorem 5.1. *The dimension N of the inertial manifold given by Theorem 4.1 is bounded as follows:*

$$N \leq c[G_0^5 G_4^5 + G_0^{60}]^2. \tag{5.21}$$

Remark 5.1. Let us mention here the result in [EFNT] concerning exponential attractors. Exponential attractors are (nonsmooth) positively invariant sets that attract all orbits at an exponential rate. For general dissipative equations, and in particular for the 2D Navier-Stokes equations with general boundary conditions, existence of such sets was shown in [EFNT]; furthermore the upper estimate of their dimension can be the same as that of the attractor, i.e. cG_0. For the equation we consider, the dimension of the exponential attractor is much smaller than that of the inertial manifold; however it is not a smooth manifold.

□

6. Complementary Results

6.1. Remarks on the Space Periodic Case.

We briefly present here a simplified proof of the result of [K1] on the existence of inertial forms for the 2D Navier-Stokes equations with periodic boundary conditions when the ratio of the periods is rational. This proof differs from [K1] in that the embedded reaction-diffusion system is much simpler and the analysis is consequently simplified.

Let $x = (x^1, x^2)$ be the coordinate for the $2D$ torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. We consider the $2D$ Navier–Stokes equations on $\mathbb{T}^2$:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p - \nu\Delta u = f, \\ \text{div } u = 0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{6.1}$$

where $u = u(x,t) = (u^1, u^2)$ is the velocity field and $p = p(x,t)$ is the pressure.

We define as usual the following function spaces (cf. [T1]):

$$\begin{cases} H_0 = \{u \in L^2(\mathbb{T}^2)^2 | \int_{\mathbb{T}^2} u dx^1 dx^2 = 0, \text{div } u = 0, \} \\ V_0 = H_0 \cap H^1(\mathbb{T}^2)^2, \\ H_1 = (H_0)^2, H_2 = L^2(\mathbb{T}^2)^4, \\ V_1 = (V_0)^2, V_2 = H^1(\mathbb{T}^2)^4, \\ H = H_0 \times H_1 \times H_2, \\ V = V_0 \times V_1 \times V_2. \end{cases} \tag{6.2}$$

We also consider the orthogonal projectors:

$$\begin{cases} P_0 : L^2(\mathbb{T}^2)^2 \to H_0, \\ P_1 : L^2(\mathbb{T}^2)^4 \to H_1, \\ P : L^2(\mathbb{T}^2)^8 \to H. \end{cases} \tag{6.3}$$

Now we define an injection map $J : V_0 \to H$ by setting

$$J(u) = (u, \tilde{v}, \tilde{w}), \tag{6.4}$$

where

$$\begin{cases} \tilde{v} = \nabla u = u^i_{;j} e_i \otimes e^j, \\ \tilde{w} = u \otimes u = u^i u^j e_i \otimes e_j, \end{cases}$$

$e_i (i = 1, 2)$ being the unit vectors in x^1 and x^2 directions respectively, with dual basis denoted by $\{e^1, e^2\}$.

Then we can consider the following embedded system

$$\begin{cases} u_t = \nu A_0 u + P_0(w^{ik}_{;k} e_i) = f, \\ v_t + \nu A_1 v + P_1[w^{ik}_{;kj} e_i \otimes e_j] = \nabla f, \\ w_t + \nu A_2 w + u \otimes P_0 Tr(u \otimes v) + P_0 Tr(u \otimes v) \otimes u \\ \qquad + 2\nu \displaystyle\sum_{k=1}^{2} v^i_k v^j_k e_i \otimes e_j - (f \otimes u + u \otimes f) \\ \qquad + \underline{k\nu[1 + \nu^{-4} \|u\|^4_{H^{3/4}} + \nu^{-2}(\|u\|^2_{H^{3/2}} + \|v\|^2_{H^{1/2}})](w - u \otimes u)} = 0. \end{cases} \tag{6.5}$$

It is easy to see that $\{u, v = \tilde{v}, w = \tilde{w}\}$ satisfy equations (6.5), the underlined terms in (6.5) being equal to 0. Considering then the system (6.5), we can prove exactly as we did for the NSE on the sphere S^2, that for k large enough, (6.5) is a dissipative system. Then all results in [K1] can be proved. Especially, there is a global attractor $\mathcal{A}$ for the embedded system (6.5) given by

$$\mathcal{A} = J(\mathcal{A}_0), \tag{6.6}$$

$\mathcal{A}_0$ being the global attractor of the original Navier–Stokes equations (6.1). Moreover, by studying the prepared system of (6.5) as (4.4) one can prove the existence of inertial manifolds for the prepared system of (6.5). As a consequence, there is an inertial form of (6.1), which reproduces all the dynamics of the Navier–Stokes equations (6.1).

6.2. The Barotropic Equations of the Atmosphere.

All our previous results are also true after some modifications for the following $2D$ barotropic vorticity equation of the globe atmosphere, which was derived by Rossby in the 1930s, and was used by J.G. Charney, R. Fjørtoft & J. von Neumann [CFN] in the first weather predictions in the 1950s:

$$\begin{cases} u_t + \nabla_u u + 2\Omega\cos\theta k \times u - \nu\triangle u = f, \\ u\big|_{t=0} = u_0, \end{cases} \tag{6.7}$$

where the space domain is S_a^2, the 2D sphere with radius a, Δ is the Laplace–Beltrami operator on S_a^2, k is the unit outward normal vector, Ω is the angular velocity of the earth, and θ is the colatitude of the earth. The existence and dimension estimates of attractors of these equations (in vorticity form) were studied in [W]. We remark here that all the results in Sections 1–5 hold true for equations (6.7). Particularly, we can obtain an inertial form of (6.7) with dimension given by an estimate similar to (5.21).

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